Carry
(PhD Version)

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Roadmap

What is Carry?

- Consider a security/commodity with spot price S_t:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t' dW_t \tag{1}$$

(μ_t, σ_t adapted to filtration $\mathcal{F}_t$ generated by W_t)

- An SDF Λ_t prices the security/commodity, has dynamics

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \lambda_t' dW_t, \tag{2}$$

risk-neutral measure $\mathbb{Q}$, $dW_t^{\mathbb{Q}} = \lambda_t dt + dW_t$ (r_t, λ_t adapted to $\mathcal{F}_t$)

- Let $F_t(\tau)$ denote futures price at t for contract expiring at $t+\tau$
- **Defn.** *Carry* is the futures return on the "front" ($\tau \downarrow 0$) contract if the spot price doesn't change ($F_{t+\tau}(0) = S_{t+\tau} = S_t$):

$$c_t := \lim_{\tau \downarrow 0} \frac{1}{\tau} \frac{S_t - F_t(\tau)}{F_t(\tau)} \tag{3}$$

(continuous-time limit of definition used by Koijen et al. (2018))

What Does Carry Measure?

- **Prop.** Carry is the decay of the spot under $\mathbb{Q}$:

$$\boxed{c_t = -\mu_t^{\mathbb{Q}} = \lambda_t' \sigma_t - \mu_t = \text{risk premium} - \text{spot drift}} \tag{4}$$

(idea: carry can't distinguish compensation for risk from drift of spot)

- **Pf.** Write the spot at $t+\tau$ as

$$S_{t+\tau} = S_t + \int_t^{t+\tau} S_u \mu_u^{\mathbb{Q}} du + \int_t^{t+\tau} S_u \sigma_u' dW_u^{\mathbb{Q}} du \tag{5}$$

using change of measure

- $\mathbb{E}_t^{\mathbb{Q}}[d0 + dF_t(\tau)] = 0 \cdot r_t dt$, $F_t(0) = S_t$, and hence $F_t(\tau) = \mathbb{E}_t^{\mathbb{Q}}[S_{t+\tau}]$:

$$\lim_{\tau \downarrow 0} \frac{1}{\tau}(F_t(\tau) - S_t) = \lim_{\tau \downarrow 0} \mathbb{E}_t^{\mathbb{Q}} \left[\frac{1}{\tau} \int_t^{t+\tau} S_u \mu_u^{\mathbb{Q}} du \right] = S_t \mu_t^{\mathbb{Q}} \tag{6}$$

from which obtains $c_t = -\mu_t^{\mathbb{Q}}$

- $\mu_t^{\mathbb{Q}} = \mu_t - \lambda_t' \sigma_t$ follows again from change of measure ∎

A Note on FX

- Two currencies i, j, with SDFs $\Lambda_{i,t}$, $\Lambda_{j,t}$
- **Prop.** If $S_{i,j,t}$ is the price of j in i ($S_{\mathrm{USD},\mathrm{JPY},t}$ is the JPY/USD), then

$$\frac{S_{i,j,t}}{S_{i,j,0}} = \frac{\Lambda_{j,t}/\Lambda_{j,0}}{\Lambda_{i,t}/\Lambda_{i,0}} \tag{7}$$

- **Pf.** Let $R_{j,t}$ denote FV of one unit of j compounded at j's short rate:

$$R_{j,t} = e^{\int_0^t r_{j,s}\,ds} \tag{8}$$

- Must price *on every event A* (Arrow–Debreu argument):

$$\mathbb{E}_0\left[\frac{\Lambda_{i,t}}{\Lambda_{i,0}} \cdot \frac{S_{i,j,t}}{S_{i,j,0}} R_{j,t}\mathbf{1}_A\right] = 1 = \mathbb{E}_0\left[\frac{\Lambda_{j,t}}{\Lambda_{j,0}} \cdot R_{j,t}\mathbf{1}_A\right] \qquad \blacksquare \tag{9}$$

- **Coro.** $S_{i,j,t}$ has $\mathbb{Q}_i$ dynamics

$$\frac{dS_{i,j,t}}{S_{i,j,t}} = \left(r_{i,t} - r_{j,t}\right) dt + \left(\lambda_{i,t} - \lambda_{j,t}\right)' dW_t^{\mathbb{Q}_i} \tag{10}$$

and hence $\mathbb{Q}_i$-decay is $\boxed{c_t = r_{j,t} - r_{i,t} = \text{rate differential}}$

Constant Maturity

- In most applications, we want to keep term fixed (e.g., "the 10-year")
- Consider asset at t that matures at T with price $P_{t,T}$, cash flow $D_{t,T}dt$
- Define fixed-τ price $\bar{P}_t(\tau) := P_{t,t+\tau}$, cash flow $\bar{D}_t(\tau) := D_{t,t+\tau}$
- Note: $dP_{t,t+\tau} = d\bar{P}_t(\tau) - \partial_\tau \bar{P}_t(\tau)dt$
- Define *forward rate* $f_t(\tau) := -\partial_\tau \log \bar{P}_t(\tau)$, *yield* $d_t(\tau) := \bar{D}_t(\tau)/\bar{P}_t(\tau)$
- Consider fixed-τ account with balance V_t:

$$dV_t = N_t(d\bar{P}_t(\tau) + D_t(\tau)dt), \tag{11}$$

where $N_t = V_t/\bar{P}_t(\tau)$ (always rebalancing so we are in the τ-security), hence

$$\frac{dV_t}{V_t} = \frac{d\bar{P}_t(\tau) - \partial_\tau \bar{P}_t(\tau)dt + D_t(\tau)dt}{\bar{P}_t(\tau)} = \frac{d\bar{P}_t(\tau)}{\bar{P}_t(\tau)} + f_t(\tau)dt + d_t(\tau)dt \tag{12}$$

- V_t must have a $\mathbb{Q}$-drift of $r_t dt$, therefore $\bar{P}_t$ has a $\mathbb{Q}$-decay of

$$\boxed{c_t = f_t(\tau) + d_t(\tau) - r_t = \text{roll} + \text{cash flow} - \text{financing}} \tag{13}$$

A Note on Futures

- Sometimes futures are the "underlying" (e.g., commodities, volatility)
- Consider futures contract at t that expires at T with futures price $F_{t,T}$
- Define $\bar{F}_t(\tau) := F_{t,t+\tau}$, note $dF_{t,t+\tau} = d\bar{F}_t(\tau) - \partial_\tau \bar{F}_t(\tau)dt$
- Consider fixed-τ account with balance V_t:

$$dV_t = V_t r_t dt + N_t dF_{t,t+\tau}, \tag{14}$$

where $N_t = V_t / \bar{F}_t(\tau)$ (always rebalancing so we are in the τ-contract), hence

$$\frac{dV_t}{V_t} = r_t dt + \frac{d\bar{F}_t - \partial_\tau \bar{F}_t(\tau)dt}{\bar{F}_t(\tau)} = \frac{d\bar{F}_t(\tau)}{\bar{F}_t(\tau)} + r_t dt + u_t(\tau)dt \tag{15}$$

where $u_t(\tau) := -\partial_\tau \log \bar{F}_t(\tau)$

- V_t has $\mathbb{Q}$-drift $r_t dt$, therefore $\bar{F}_t$ has $\mathbb{Q}$-decay $\boxed{c_t = u_t(\tau) = \text{roll yield}}$
- The *basis* is $S_t - F_t(\tau)$; futures curve said to be in
  - *backwardation* if $F_t(\tau) < S_t$ (negative basis, positive average carry) or
  - *contango* if $F_t(\tau) > S_t$ (positive basis, negative average carry)

A Markov Model

- So far only assumed μ_t, σ_t are $\mathcal{F}_t$-adapted — now, add structure
- Let $X_t \in \mathbb{R}^N$ be a Markov state (growth, inflation, etc.), adapted to $\mathcal{F}_t$:

$$dX_t = m(X_t)dt + \Sigma(X_t)dW_t \tag{16}$$

- **Conj.** Fixed-τ price $\bar{P}_t(\tau) = F(\tau, X_t)$, dividend $\bar{D}_t(\tau) = G(\tau, X_t)$.
- $m^{\mathbb{Q}}(X_t) := m(X_t) - \Sigma(X_t)\lambda_t$ (change of measure)
- Define *duration* $\beta(\tau, X) := -F_X/F$ and *convexity* $\mathcal{C}(\tau, X) := F_{XX}/F$
- Itô on $\bar{P}_t(\tau) = F(\tau, X_t)$ at fixed τ:

$$\frac{d\bar{P}_t(\tau)}{\bar{P}_t(\tau)} = \Big[-\beta' m^{\mathbb{Q}} + \frac{1}{2}\mathrm{tr}(\mathcal{C}\Sigma\Sigma')\Big]dt - \beta'\Sigma\, dW_t^{\mathbb{Q}} \tag{17}$$

from which carry obtains:

$$\boxed{c_t = \beta' m^{\mathbb{Q}} - \tfrac{1}{2}\mathrm{tr}(\mathcal{C}\Sigma\Sigma') = \text{duration} \times \text{drift} - \tfrac{1}{2}\text{convexity} \times \text{var}} \tag{18}$$

An Affine Example

- Let X_t follow OU-process:

$$dX_t = \kappa(\mu - X_t)dt + \Sigma dW_t \tag{19}$$

with $\kappa, \Sigma \in \mathbb{R}^{N\times N}$, $\mu \in \mathbb{R}^N$, $W_t \in \mathbb{R}^N$

- *Example:* $X_t = r_t$, $dr_t = \kappa(\mu - r_t)dt + \sigma dW_t$ (i.e., *Vasicek* model)
- SDF Λ_t evolves according to

$$\frac{d\Lambda_t}{\Lambda_t} = -r(X_t)dt - \lambda(X_t)' dW_t, \tag{20}$$

short rate $r(X_t) = \delta_0 + \delta_1' X_t$, *price of risk* $\lambda(X_t) = \lambda_0 + \lambda_1' X_t$

- Consider a claim to the dividend D_T on date T, where

$$\frac{dD_t}{D_t} = g(X_t)dt, \tag{21}$$

growth rate $g(X_t) = \gamma_0 + \gamma_1' X_t$

An Affine Example

- Let $\tau = T - t$ and use $0 = \mathbb{E}_t[d(\Lambda_t P_t(\tau))]$ to get the price

$$P_t(\tau) = D_t e^{\alpha(\tau) - \beta(\tau)' X_t} \tag{22}$$

where α and β satisfy $\alpha(0) = \beta(0) = 0$ and

$$\alpha'(\tau) = -(\delta_0 - \gamma_0) - \beta(\tau)'(\kappa\mu - \Sigma\lambda_0) + \frac{1}{2}\beta(\tau)'\Sigma\Sigma'\beta(\tau) \tag{23}$$

$$\beta'(\tau) = (\delta_1 - \gamma_1) - (\kappa + \Sigma\lambda_1)'\beta(\tau) \tag{24}$$

- Using equations (23) and (24), we have

$$c_t = \underbrace{\sigma_t(\tau)'\lambda(X_t)}_{\text{risk premia}} - \underbrace{\beta(\tau)'\kappa(X_t - \mu)}_{\text{mean reversion}} - \frac{1}{2}\underbrace{\beta(\tau)'\Sigma\Sigma'\beta(\tau)}_{\text{convexity}} - \underbrace{g(X_t)}_{\text{growth}} \tag{25}$$

where $\sigma_t(\tau) = -\Sigma'\beta(\tau)$

- Equation (4), but spot drift = mean reversion + convexity + growth

Some History

- Results from Koijen, Moskowitz, Pedersen, and Vrugt (2018)
- They construct a discrete-time measure C_t^i of carry
- Main regression:

$$r_{t+1}^i = a^i + b_t + cC_t^i + \epsilon_{t+1}^i \tag{26}$$

  - $c < 0$: positive carry $\Rightarrow$ negative returns
  - $c = 0$: returns are unpredictable
  - $c \in (0, 1)$: positive carry $\Rightarrow$ positive returns, price depreciation
  - $c = 1$: prices changes are unpredictable
  - $c > 1$: positive carry $\Rightarrow$ price appreciation

Some History

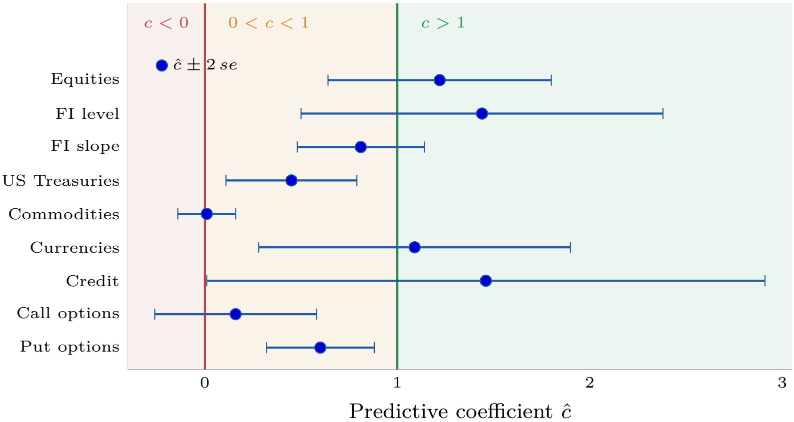

Table 4 of Koijen, Moskowitz, Pedersen, and Vrugt (2018).

Conclusion

Takeaways

- Carry is
  - return on front-month futures if spot doesn't change (*observable*)
  - roll yield + cash flow yield − financing (*observable*)
  - risk premium − spot drift (*unobservable*)
  - duration × drift $-\frac{1}{2}$ convexity × variance (*unobservable*)
- High carry ⇒ high price of risk or low expectations? Role for analyst
- Carry has historically predicted excess returns (future?)

Other Musings

- Investing is about the future, not the past
- Estimating risk loadings/prices (i.e., $\mathbb{E}_t[R^e_{t+1}] = \beta_t' \lambda_t$) is a fool's errand
- Carry is our lodestar — it is the only thing truly observable

References

Koijen, R. S. J., Moskowitz, T. J., Pedersen, L. H., and Vrugt, E. B. (2018). Carry. *Journal of Financial Economics*, 127(2):197–225.